

Probabilistic model of N correlated binary random variables and non-extensive statistical mechanics

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The framework of non-extensive statistical mechanics, proposed by Tsallis, has been used to describe a variety of systems. The non-extensive statistical mechanics is usually introduced in a formal way, thus simple models exhibiting some important properties described by the non-extensive statistical mechanics are useful to provide deeper physical insights. In this article we present a simple model, consisting of a one-dimensional chain of particles characterized by binary random variables, that exhibits both the extensivity of the generalized entropy with $q < 1$ and a q -Gaussian distribution in the limit of the large number of particles.

I. INTRODUCTION

There exist a number of systems featuring long-range interactions, long-range memory, and anomalous diffusion, that possess anomalous properties in view of traditional Boltzmann-Gibbs statistical mechanics. Non-extensive statistical mechanics is intended to describe such systems by generalizing the Boltzmann-Gibbs statistics [1–3]. In general, the non-extensive statistical mechanics can be applied to describe the systems that, depending on the initial conditions, are not ergodic in the entire phase space and may prefer a particular subspace which has a scale invariant geometry, a hierarchical or multifractal structure. Concepts related to the non-extensive statistical mechanics have found applications in a variety of disciplines: physics, chemistry, biology, mathematics, economics, and informatics [4–6].

The non-extensive statistical mechanics is based on a generalized entropy [1]

$$S_q = \frac{1 - \int [p(x)]^q dx}{q - 1}, \quad (1)$$

where $p(x)$ is a probability density function of finding the system in the state characterized by the parameter x , while q is a parameter describing the non-extensiveness of the system. Entropy (1) is an extension of the Boltzmann-Gibbs entropy

$$S_{BG} = - \int p(x) \ln p(x) dx \quad (2)$$

which is recovered from Eq. (1) in the limit $q \rightarrow 1$ [1, 2]. More generalized entropies and distribution functions are introduced in Refs. [7, 8]. Statistics associated to Eq. (1) has been successfully applied to phenomena with the scale-invariant geometry, like in low-dimensional dissipative and conservative maps in the dynamical systems [9–11], anomalous diffusion [12, 13], turbulent flows [14], Langevin dynamics with fluctuating temperature [15, 16], spin-glasses [17], plasma [18] and to the financial systems [19–21].

By maximizing the entropy (1) with the constraints $\int_{-\infty}^{+\infty} p(x) dx = 1$ and

$$\frac{\int_{-\infty}^{+\infty} x^2 [p(x)]^q dx}{\int_{-\infty}^{+\infty} [p(x)]^q dx} = \sigma_q^2, \quad (3)$$

where σ_q^2 is the generalized second-order moment [22–24], one obtains the q -Gaussian distribution density

$$p_q(x) = C \exp_q(-A_q x^2). \quad (4)$$

Here $\exp_q(\cdot)$ is the q -exponential function, defined as

$$\exp_q(x) \equiv [1 + (1 - q)x]_+^{\frac{1}{1-q}}, \quad (5)$$

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with $[x]_+ = x$ if $x > 0$, and $[x]_+ = 0$ otherwise. The q -Gaussian distribution (or distribution very close to it) appears in many physical systems, such as cold atoms in dissipative optical lattices [25], dusty plasma [18], motion of hydra cells [26], and defect turbulence [27]. The q -Gaussian distribution is one of the most important distributions in the non-extensive statistical mechanics. Its importance stems from the generalized central limit theorems [28–30]. According to q -generalized central limit theorem, q -Gaussian can result from a sum of N q -independent random variables. The q -independence is defined in [28] through the q -product [31, 32], and the q -generalized Fourier transform [28]. When $q \neq 1$, q -independence corresponds to a global correlation of the N random variables. However, the rigorous definition of q -independence is not transparent enough in physical terms.

The non-extensive statistical mechanics is introduced in a formal way, starting from the maximization of the generalized entropy [1]. Therefore, simple models providing some degree of intuition about non-extensive statistical mechanics can be useful for understanding it. There has been some effort to create such simple models. In Ref. [33] a system composed of N distinguishable particles, each particle characterized by a binary random variable, has been constructed so that the number of states with non-zero probability grows with the number of particles N not exponentially, but as a power law. For such a system in the limit $N \rightarrow \infty$ the ratio $S_q(N)/N$ is finite not for the Boltzmann-Gibbs entropy but for the generalized entropy with some specific value of q . The starting point in the construction is the Leibnitz triangle, then initial probabilities are redistributed into a small number of all the other possible states, in such a way that the norm is preserved. For example, in the restricted uniform model [33] for a fixed value of N all nonvanishing probabilities are equal. In the proposed models that yield $q \neq 1$ there are $d + 1$ no-zero probabilities and the value of q is given by $q = 1 - 1/d$.

In Refs. [34, 35], the goal has been to construct simple models providing q -Gaussian distributions. As in [33], the models considered in [34] consist of N independent and distinguishable binary variables, each of them having two equally probable states. The models presented in [34] are strictly scale-invariant, however, they do not approach a q -Gaussian form when the number of particles N in the model increases [36]. The situation is different with the models presented in [35]: the two proposed models do approach a q -Gaussian form, the second of them does so by construction. All models in [34, 35], except the last model of [35] are for $q \leq 1$. The drawback of the models from Ref. [35] is that the standard Boltzmann-Gibbs entropy remains extensive. In addition, the models are constructed artificially and it is hard to see how they can be related to real physical systems.

The goal of this paper is to provide a simple model that achieves both the extensivity of the generalized entropy with $q \neq 1$ and q -Gaussian distribution in the limit of the large number of particles. In addition, we want to construct a model that is closer to situations in physical systems. We expect that such a model can provide deeper insights into non-extensive statistical mechanics than the previously constructed simple models.

The paper is organized as follows: To highlight differences from our proposed model, a simple model consisting of uncorrelated binary random variables and leading to extensive Boltzmann-Gibbs entropy and a Gaussian distribution is presented in Section II. In Section III we construct a simple model exhibiting the extensivity of the generalized entropy with $q \neq 1$ and q -Gaussian distribution in the limit of the large number of particles. Section IV summarizes our findings.

II. MODEL OF UNCORRELATED BINARY RANDOM VARIABLES

At first let us consider a model consisting from N uncorrelated binary random variables. Physical implementation of such a model could be N particles of spin $\frac{1}{2}$, the projection of each spin to the z axis can acquire the values $\pm\frac{1}{2}$. The microscopic configuration of the system can be described by a sequence of spin projections $s_1 s_2 \dots s_N$, where each $s_i = \pm\frac{1}{2}$. There are $W = 2^N$ different microscopic configurations. As is usual in statistical mechanics for the description of a microcanonical ensemble, we assign to each microscopic configuration the same probability. Thus the probability of each microscopic configuration is

$$P = \frac{1}{W} = \frac{1}{2^N}. \quad (6)$$

Note, that this system has a property of composability: if we have two spin chains with W_1 and W_2 microscopic configurations, then we can join them to form a larger system. The description of a larger system is just concatenation of the descriptions of each subsystems and the number of microscopic configurations of the whole system is $W = W_1 W_2$. The standard Boltzmann-Gibbs entropy $S_{\text{BG}} = k_B \ln W$ is extensive for this system: $S_{\text{BG}} = N k_B \ln 2$ grows linearly with N .

Let us consider a macroscopic quantity, the total spin of the system

$$M = \sum_{i=1}^N s_i. \quad (7)$$

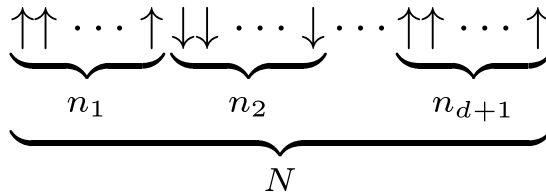


FIG. 1. One-dimensional spin chain having N spins. There are d spin flips, so that the spin chain consists of $d + 1$ domains of spins pointing to the same direction. The lengths of the domains are n_1, n_2, \dots, n_{d+1} .

The total spin can take values $M = -\frac{N}{2}, -\frac{N}{2} + 1, \dots, \frac{N}{2} - 1, \frac{N}{2}$. The value of M can be obtained when there are $n = M + \frac{N}{2}$ spins with the projection $+\frac{1}{2}$, the remaining spins have projection $-\frac{1}{2}$. The macroscopic configuration corresponding to the given value of M can be realized by $\frac{N!}{n!(N-n)!}$ microscopic configurations, thus using Eq. (6) the probability of each macroscopic configuration is

$$P_M = \frac{1}{2^N} \frac{N!}{n!(N-n)!}. \quad (8)$$

Note, that the probabilities of macroscopic configurations are normalized:

$$\sum_{M=-N}^N P_M = 1. \quad (9)$$

Using Eq. (8) we can calculate the average spin of the system $\langle M \rangle = 0$ and the standard deviation

$$\sqrt{\langle M^2 \rangle - \langle M \rangle^2} = \frac{\sqrt{N}}{2}. \quad (10)$$

From Eq. (10) it follows that the relative width of the distribution of the total spin M decreases with number of spins N as $\frac{1}{\sqrt{N}}$. When N is large then we can approximate the factorials using Stirling formula

$$n! \approx \sqrt{2\pi n} n^n e^{-n} \quad (11)$$

and obtain a Gaussian distribution

$$P_M \approx \frac{1}{\sqrt{\pi \frac{N}{2}}} e^{-\frac{2M^2}{N}} \quad (12)$$

The Gaussian distribution can be obtained by maximizing the Boltzmann-Gibbs entropy (2) with appropriate constraints.

III. MODEL OF CORRELATED SPINS

In this Section we investigate a system consisting from N correlated binary random variables. Similarly as in the previous Section, we can think about a one-dimensional spin chain consisting from N spins. However, the spins are correlated: spins next to each other have almost always the same direction, except there are d cases when the next spin has an opposite direction, as is shown in Fig. 1. Thus the spin chain consists from $d + 1$ domains with spins pointing in the same direction and has d boundaries between domains. This model is inspired by a connection between non-extensive statistics and critical phenomena [37–39]. As it has been shown in Refs. [37–39], the properties of a single large cluster of the order parameter at a critical point in thermal systems can be described by non-extensive statistics. The restriction of the number of allowed states in our model is very similar to the models presented in [33].

Let us calculate the number of allowed microscopic configurations of the spin chain. If the length of i -th domain is n_i then we need to calculate the number of possible partitions such that

$$\sum_{i=1}^{d+1} n_i = N. \quad (13)$$

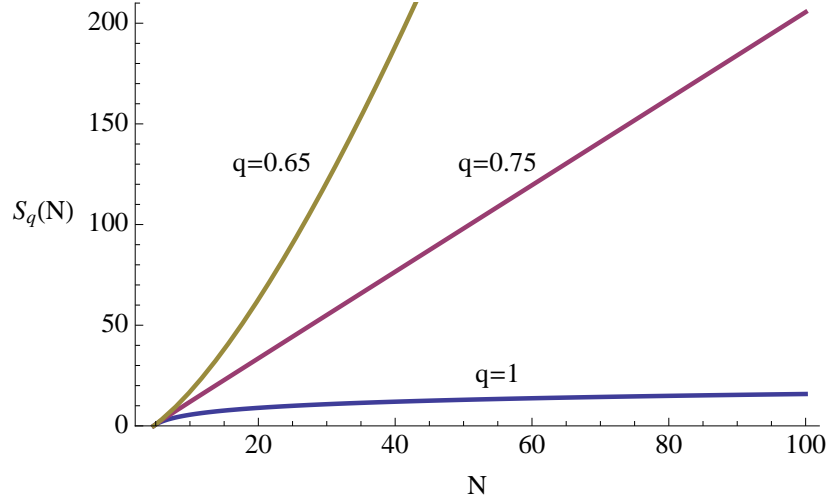


FIG. 2. Dependence of the generalized entropy S_q on the number of particles N in the spin chain for various values of q . The number of domain walls $d = 4$. Only for $q = q_{\text{stat}} = 1 - 1/d = 0.75$ the limit $\lim_{N \rightarrow \infty} S_q(N)/N$ has a finite value; the limit vanishes (diverges) for $q > 0.75$ ($q < 0.75$).

This number is equivalent to the number of ways one can place d domain boundaries into $N - 1$ possible positions. Since the spins in the first domain can be up or down, the number of microscopic configurations is twice as large. Thus the number of allowed microscopic configurations of the spin chain is

$$W = \frac{2(N-1)!}{d!(N-d-1)!} = \frac{2}{d!}(N-1) \cdots (N-d). \quad (14)$$

For large $N \gg d$ we have that the number of microscopic configurations grows as a power-law of the number of spins, not exponentially:

$$W \sim \frac{2}{d!} N^d. \quad (15)$$

Assigning to each allowed microscopic configuration the same probability $P = 1/W$ we get that in this situation the traditional Boltzmann-Gibbs entropy is not linearly proportional to N and thus is not extensive. The generalized entropy Eq. (1) for equal probabilities $1/W$ takes the form

$$S_q = k_B \frac{1 - W^{1-q}}{q-1}. \quad (16)$$

Using Eqs. (15) and (16) one gets that the generalized entropy is extensive (proportional to N) only when q is

$$q_{\text{stat}} = 1 - \frac{1}{d}. \quad (17)$$

This is the same dependency of q on d as in [33].

In Fig. 2 the dependence of the generalized entropy (16) on the size N of the spin chain for various values of q is shown. As one can see, only for $q = q_{\text{stat}}$ the limit $\lim_{N \rightarrow \infty} S_q/N$ is finite, this limit vanishes or diverges for all other values of q . Thus our simple model can be characterized by the generalized entropy with $q < 1$.

Now let us consider the macroscopic variable, the total spin M . If there are $d + 1$ domains and the total spin is M then we have the equality

$$s_1 \sum_{i=1}^{d+1} (-1)^{i-1} n_i = M. \quad (18)$$

Eq. (18) can be written as

$$N_{\text{odd}} - N_{\text{even}} = \frac{M}{s_1}, \quad (19)$$

where

$$N_{\text{odd}} = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} n_{1+2k} \quad (20)$$

is the total length of odd-numbered domains,

$$N_{\text{even}} = \sum_{k=1}^{\lfloor \frac{d+1}{2} \rfloor} n_{2k} \quad (21)$$

is the total length of even-numbered domains. Here $\lfloor \cdot \rfloor$ denotes an integer part of a number. In addition, the total lengths of odd- and even-numbered domains should obey the equation

$$N_{\text{odd}} + N_{\text{even}} = N. \quad (22)$$

Eqs. (19) and (22) have only one solution

$$N_{\text{odd}} = \frac{1}{2} \left(N + \frac{M}{s_1} \right), \quad N_{\text{even}} = \frac{1}{2} \left(N - \frac{M}{s_1} \right). \quad (23)$$

The total length of odd-numbered domains N_{odd} is a sum of $\lfloor \frac{d}{2} \rfloor + 1$ terms, the total length of even-numbered domains N_{even} is a sum of $\lfloor \frac{d+1}{2} \rfloor$ terms. Similarly as in Eq. (14), the number of ways to choose n_1, n_2, \dots, n_{d+1} is equal to the number of ways to place $\lfloor \frac{d}{2} \rfloor$ domain boundaries into $N_{\text{odd}} - 1$ positions multiplied by the number of ways to place $\lfloor \frac{d+1}{2} \rfloor - 1$ domain boundaries into $N_{\text{even}} - 1$ positions. Thus the number of ways to choose the domain lengths when M , N and s_1 are given is

$$W(N, M, s_1) = \frac{1}{(\lfloor \frac{d}{2} \rfloor)!} \left(N_{\text{odd}} - 1 \right) \cdots \left(N_{\text{odd}} - \left\lfloor \frac{d}{2} \right\rfloor \right) \frac{1}{(\lfloor \frac{d+1}{2} \rfloor)!} \left(N_{\text{even}} - 1 \right) \cdots \left(N_{\text{even}} - \left\lfloor \frac{d-1}{2} \right\rfloor \right). \quad (24)$$

The number of microscopic configurations corresponding to a given value of M is

$$W(N, M) = \sum_{s_1 = \pm 1/2} W(N, M, s_1). \quad (25)$$

Using Eqs. (23)–(25) we obtain that the number of microscopic configurations corresponding to a given value of M is

$$W(N, M) = \frac{1}{2^{d-2} \left[\left(\frac{d-1}{2} \right)! \right]^2} \left((N-2)^2 - 4M^2 \right) \cdots \left((N-(d-1))^2 - 4M^2 \right) \quad (26)$$

when d is odd and

$$W(N, M) = \frac{(N-d)}{2^{d-2} \frac{d}{2} \left[\left(\frac{d}{2} - 1 \right)! \right]^2} \left((N-2)^2 - 4M^2 \right) \cdots \left((N-(d-2))^2 - 4M^2 \right) \quad (27)$$

when d is even. When $N \gg d$ then Eqs. (26) and (27) can be approximated as

$$W(N, M) \approx \frac{N^{d-1}}{2^{d-2} (\lfloor \frac{d}{2} \rfloor)! (\lfloor \frac{d-1}{2} \rfloor)!} \left(1 - \frac{4M^2}{N^2} \right)^{\lfloor \frac{d-1}{2} \rfloor}. \quad (28)$$

Instead of the total spin M it is convenient to consider a scaled variable

$$x = \frac{2M}{N}. \quad (29)$$

From Eq. (28) we get that the distribution $P_x(x)$ of the variable x for large N is proportional to

$$P_x(x) \propto (1 - x^2)^{\lfloor \frac{d-1}{2} \rfloor}. \quad (30)$$

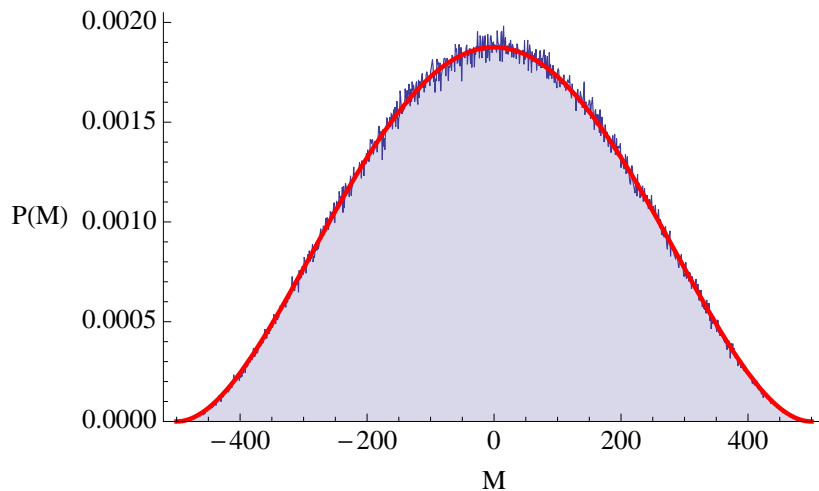


FIG. 3. Comparison of numerically obtained histogram of total spin M (gray area) with q -Gaussian distribution (solid red line). The histogram is calculated using an ensemble of randomly generated spin chains that have the length $N = 1000$ spins and $d = 5$ spin flips. The q -Gaussian distribution is given by Eq. (32) with $x_q = N/2$.

When $q < 1$ then the q -Gaussian distribution (4) has compact support, the range of possible values of x is limited by the condition $|x| \leq x_q$, where

$$x_q = \frac{1}{\sqrt{(1-q)A_q}}. \quad (31)$$

Using the limiting value x_q the expression (4) for the q -Gaussian distribution takes the form

$$p_q(x) = \frac{\Gamma\left(\frac{5-3q}{2(1-q)}\right)}{\sqrt{\pi}x_q\Gamma\left(\frac{2-q}{1-q}\right)} \left[1 - \frac{x^2}{x_q^2}\right]_+^{\frac{1}{1-q}}. \quad (32)$$

By rescaling the variable x the expression (32) for q -Gaussian distribution in the case of $q < 1$ can be written as

$$p_q(x) \propto (1 - x^2)^{\frac{1}{1-q}}. \quad (33)$$

Comparing Eq. (30) with Eq. (33) we see that the distribution of the total spin M in our model in the limit of large number of spins N is a q -Gaussian with

$$q_{\text{dist}} = 1 - \frac{1}{\left[\frac{d-1}{2}\right]}. \quad (34)$$

Comparison of the histogram of the total spin M , calculated using an ensemble of randomly generated spin chains, with a q -Gaussian distribution (32) is shown in Fig. 3. We see that the q -Gaussian describes the distribution of the total spin very well. By increasing the number of spin flips d , both q_{stat} and q_{dist} approaches the value of 1, as follows from Eqs. (17) and (34). In the limit of $d \rightarrow \infty$ the distribution of total spin becomes Gaussian.

From Eqs. (17) and (34) follows the relationship between the two q values:

$$\frac{1}{1 - q_{\text{dist}}} = \left\lfloor \frac{1}{2} \left(\frac{1}{1 - q_{\text{stat}}} - 1 \right) \right\rfloor. \quad (35)$$

This equation is similar to relations presented in Eq. (18) of Ref. [40], in the footnote in page 15378 of Ref. [33], and Eq. (5) in Ref. [41].

IV. DISCUSSION

Our simple model of a spin chain exhibits both non-extensive behavior of Boltzmann-Gibbs entropy and q -Gaussian distribution of the total spin. This is in contrast to the other models involving N binary random variables: the

models presented in [33] demonstrate that the generalized entropy with $q \neq 1$ can be extensive, but they do not provide q -Gaussian distribution, whereas the models from Ref. [35] yield q -Gaussians but for them the Boltzmann-Gibbs entropy is extensive. Thus the model, presented in this paper, is an improvement over earlier models and may provide deeper insights into non-extensive statistical mechanics.

By comparing the model with the chain of uncorrelated spins, presented in Section II, we see that the reason for the non-extensivity of Boltzmann-Gibbs entropy and the extensivity of the generalized entropy is the reduction of the number of allowed microscopic configurations, resulting from the restriction of the number of allowed spin flips. Similar reason is behind the differences in the distribution of the total spin M : the number of possible configurations having N_{odd} spins pointing in one direction and N_{even} spins in the opposite direction in the chain of uncorrelated spins is the exponential function of $N_{\text{odd}}N_{\text{even}}$, resulting in the Gaussian function of M . When the number of spin flips is restricted, the number of configurations is a power-law function of $N_{\text{odd}}N_{\text{even}}$, resulting in the q -Gaussian.

The model can be slightly modified by requiring that the number of spin flips is not constant, but can be a random number not larger than d . Since the number of microscopic configurations having d spin flips grows as N^d , the contribution of configurations with smaller number of spin flips becomes negligible in the limit of large N . Thus the conclusions of Section III remain valid also for this modified model.

Note, that the values of q obtained from the entropy (17) and from the distribution (34) are different. This difference is not surprising and is similar to the q -triplet in the non-extensive statistical mechanics. Usually the systems described by the non-extensive statistical mechanics have three different values of q , the q -triplet [1]. This triplet consist from the values of q , (q_{sen} , q_{rel} , q_{stat}) obtained from the sensitivity to the initial conditions, the relaxation in phase-space, and the distribution of energies at a stationary state [1]. Since our model does not involve the energy and there is no evolution in time, we have obtained only two values q . It would be interesting task for the future to extend the model and get the full q -triplet.

The model presented here provides q -Gaussian distribution with $q < 1$. Thus another open question is whether it is possible to modify the model to obtain q -Gaussian distribution with $q > 1$.

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- [1] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics – Approaching a Complex World* (Springer, New York, 2009).
 - [2] C. Tsallis, Braz. J. Phys. **39**, 337 (2009).
 - [3] L. Telesca, Tectonophysics **494**, 155 (2010).
 - [4] C. M. Gell-Mann and C. Tsallis, *Nonextensive Entropy—Interdisciplinary Applications* (Oxford University Press, NY, 2004).
 - [5] S. Abe, Astrophys. Space Sci. **305**, 241 (2006).
 - [6] S. Picoli, R. S. Mendes, L. C. Malacarne, and R. P. B. Santos, Braz. J. Phys. **39**, 468 (2009).
 - [7] R. Hanel and S. Thurner, EPL **93**, 20006 (2011).
 - [8] R. Hanel, S. Thurner, and M. Gell-Mann, PNAS **108**, 6390 (2011).
 - [9] O. Afsar and U. Tirnakli, EPL **101**, 20003 (2013).
 - [10] U. Tirnakli, C. Tsallis, and C. Beck, Phys. Rev. E **79**, 056209 (2009).
 - [11] G. Ruiz, T. Bountis, and C. Tsallis, Int. J. Bifurcation Chaos **22**, 1250208 (2012).
 - [12] Z. Huang, G. Su, A. El Kaabouchi, Q. A. Wang, and J. Chen, J. Stat. Mech. **2010**, L05001 (2010).
 - [13] J. Prehl, C. Essex, and K. H. Hoffman, Entropy **14**, 701 (2012).
 - [14] C. Beck and S. Miah, Phys. Rev. E **87**, 031002 (2013).
 - [15] A. A. Budini, Phys. Rev. E **86**, 011109 (2012).
 - [16] J.-L. Du, J. Stat. Mech. **2012**, P02006 (2012).
 - [17] R. M. Pickup, R. Cywinski, C. Pappas, B. Farago, and P. Fouquet, Phys. Rev. Lett. **102**, 097202 (2009).
 - [18] B. Liu and J. Goree, Phys. Rev. Lett. **100**, 055003 (2008).
 - [19] L. Borland, Quantit. Finance **12**, 1367 (2012).
 - [20] S. Drozd, J. Kwapien, P. Oswiecimka, and R. Rak, New J. Phys. **12**, 105003 (2010).
 - [21] V. Gontis, J. Ruseckas, and A. Kononovicius, Physica A **389**, 100 (2010).
 - [22] C. Tsallis, A. R. Plastino, and R. S. Mendes, Physica A **261**, 534 (1998).
 - [23] D. Prato and C. Tsallis, Phys. Rev. E **60**, 2398 (1999).
 - [24] C. Tsallis, Braz. J. Phys **29**, 1 (1999).
 - [25] P. Douglas, S. Bergamini, and F. Renzoni, Phys. Rev. Lett. **96**, 110601 (2006).
 - [26] A. Upadhyaya, J. P. Rieu, J. A. Glazier, and Y. Sawada, Physica A **293**, 459 (2001).
 - [27] K. E. Daniels, C. Beck, and E. Bodenschatz, Physica D **193**, 208 (2004).
 - [28] S. Umarov, C. Tsallis, and S. Steinberg, Milan J. Math. **76**, 307 (2008).
 - [29] S. Umarov, C. Tsallis, M. Gell-Mann, and S. Steinberg, J. Math. Phys. **51**, 033502 (2010).
 - [30] S. Umarov and C. Tsallis, in *Complexity, Metastability and Nonextensivity*, American Institute of Physics Conference Proceedings, Vol. 965, edited by S. Abe, H. Herrmann, P. Quarati, A. Rapisarda, and C. Tsallis (New York, 2007) pp. 34–42.

- [31] L. Nivonen, A. Le Mehaute, and Q. A. Wang, Rep. Math. Phys. **52**, 437 (2003).
- [32] E. P. Borges, Physica A **340**, 95 (2004).
- [33] C. Tsallis, M. Gell-Mann, and Y. Sato, Proc. Natl. Acad. Sc. USA **102**, 15377 (2005).
- [34] L. G. Moyano, C. Tsallis, and M. Gell-Mann, Europhys. Lett. **73**, 813 (2006).
- [35] A. Rodriguez, V. Schwammle, and C. Tsallis, J. Stat. Mech. **2008**, P09006 (2008).
- [36] H. J. Hilhorst and G. Schehr, J. Stat. Mech. **2007**, P06003 (2007).
- [37] A. Robledo, Phys. Rev. Lett. **83**, 2289 (1999).
- [38] A. Robledo, Mol. Phys. **103**, 3025 (2005).
- [39] A. Robledo, Int. J. Mod. Phys. B **21**, 3947 (2007).
- [40] G. Ruiz and C. Tsallis, Phys. Lett. A **376**, 2451 (2012).
- [41] A. Celikoglu, U. Tirnakli, and S. Queirós, Phys. Rev. E **82**, 021124 (2010).